NONLINEAR DYNAMICS AND CONTROL OF MULTI-BODY ELASTIC SPACECRAFT SYSTEMS

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ABSTRACT

This paper deals with the dynamics and control of multi-body systems in which their linear, rotational, and elastic motions are coupled. We model a multi-body spacecraft system as two rigid bodies that are connected by a nonlinear, elastic spring. It is assumed that the system travels in a planar orbit around a planet with a uniform gravitational field. The methodology developed in this paper leads to a simple nonlinear control law that allows us to control the coupled orbital, attitude, and elastic dynamics, and precisely maintain a prescribed tumbling trajectory. We show that complex motions like the precision tumbling of elastically vibrating spacecraft in orbit can be accomplished with considerable ease and accuracy. The proposed methodology reveals a systematic technique for controlling complex nonlinear, multi-body systems.

1. INTRODUCTION. The traditional method of analyzing spacecraft dynamics is to decouple the attitude motion of the spacecraft about its center of mass and that of the orbital motion. In general, this method works well, but for more complex space systems, such as those with massive appendages and booms, or for systems that are tethered, understanding the coupled dynamics and control becomes important, especially for precision-driven space missions. Perturbative forces on these complex systems cause them to torque, flex, compress, expand, and behave in other ways that affect the entire dynamics of the system. This requires one to study the motion of the system as a whole if high control authority is required. Some advancement has been made in recent years in trying to understand the dynamics of coupling the various motions between multi-body spacecraft systems. The station keeping, retrieval, and attitude control of tethered systems was studied in Refs. [1-3]. Studies of the coupled attitude and orbital dynamics of spacecraft were presented in Refs. [4-5]. The effect of gravitational gradient torques on the attitude behavior of a dumbbell-shaped spacecraft was studied in [6]. Orbital and attitude dynamics of a flexible spacecraft in the Jovian system including higher order gravity fields was worked on by Quadrelli, et. al. [7]. Sanyal et. al. [8-9] considered the control of orbital, attitude, and elastic motion for a dumbbell spacecraft in a planar orbit by linearizing a set of reduced equations of motion. In these previous studies, linearization and other approximations of the dynamics and/or control are involved.

In this paper we develop a method to address the complete three-dimensional, nonlinear dynamics and control problem wherein we consider a new nonlinear control methodology for a spacecraft system that accounts for its coupled orbit, attitude, and elastic dynamics. The control approach is developed using principles rooted in analytical mechanics and adopted for the control of complex multi-body systems. It is based on the explicit set of equations of motion for general constrained mechanical systems discovered by Udwadia and Kalaba [10] and further developed in [11] for the control of nonlinear dynamical systems. An important contribution of the proposed method is that the control is obtained in closed form, and can therefore be computed in real time. Furthermore, no linearization in the system's dynamics or in the control is involved. It has been shown [12-14] that the proposed methodology is easily applied to highly constrained astrodynamic problems.

The spacecraft system considered in this work is modeled using two rigid bodies (with masses m_1 and m_2 and principal inertia tensors I^1 and I^2) in a dumbbell-like structure connected by an elastic spring. It is in orbit around a central gravitating body with a uniform gravity field as shown in Fig. 1. The dumbbell spacecraft system is free to translate, rotate, and vibrate in space. Yet, relative motion between the two masses m_1 and m_2 occurs, when the spring stretches or compresses, only along a line $\mathbf{C}_1\mathbf{C}_2$ that is *fixed in direction relative to the two masses*. Thus, except for the fact that the two masses can stretch and contract along this line $\mathbf{C}_1\mathbf{C}_2$, the spacecraft structure behaves much like a rigid body. For simplicity, the unit vectors $\boldsymbol{\eta}^1$ and $\boldsymbol{\eta}^2$ are chosen to lie along the principal axes of inertia of the two masses m_1 and m_2 and are also taken to lie along the spring's direction (see Fig. 1). Although the spacecraft model chosen here for illustration purposes is a simple dumbbell structure, the control methodology that is developed is easily applied to more complex spacecraft configurations.

Section 2 of the paper describes the general control methodology, and in Sections 3 and 4 we summarize the development of the equations of motion. Following this, we give numerical simulation results for a dumbbell spacecraft system in a polar orbit around Mars where we assume a uniform gravity field. The required control such that the vibrating dumbbell shaped system *precisely* tumbles in a given, prescribed manner is obtained, and the effectiveness of the control is demonstrated. Section 5 gives our conclusions and remarks.

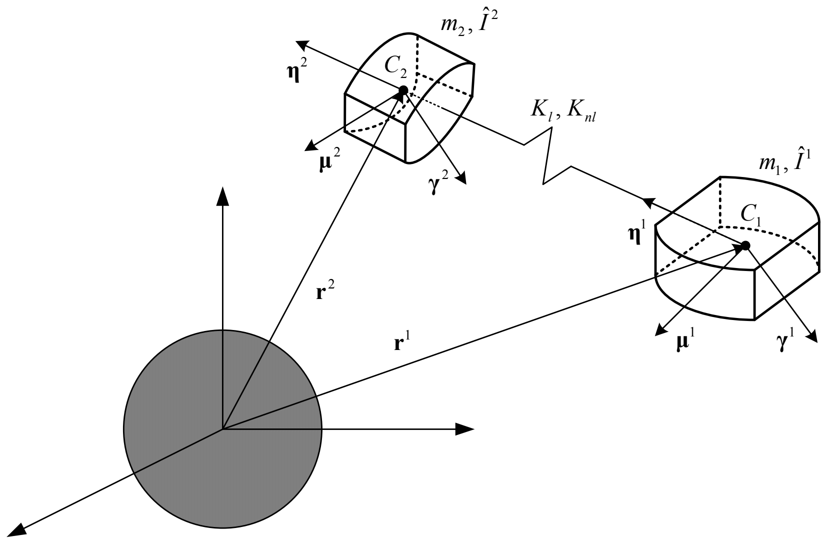

Fig. 1. Schematic of the two-mass dumbbell shaped orbiting spacecraft system. The two rigid masses m_1 and m_2 are connected by a nonlinear spring between their respective centers of mass, $\mathbf{C}_1$ and $\mathbf{C}_2$, whose direction is fixed relative to the two masses. The distance between the two masses when the spring is unstretched is l_e. K_l and K_{nl} are the spring constants for the linear and cubically nonlinear restoring forces exerted by the spring.

2. GENERAL METHODOLOGY. The development of the equations of motion for constrained mechanical systems has been pursued by numerous scientists and mathematicians including Appell [15], Dirac [16], Gauss [17], Gibbs [18], and Lagrange [19], all of whom have used the principle of virtual work or the d'Alembert-Lagrange principle as their starting point. More recently, in 1992, Udwadia and Kalaba [10] obtained a simple, *explicit*, set of equations of motion for general mechanical systems with holonomic and/or non-holonomic constraints. This section summarizes the 'fundamental equation' of motion described in [10] in the context of multi-body dynamics, which will enable us to derive a methodology to analyze and control the coupled orbital, attitude, and elastic dynamics for spacecraft systems.

We begin with a system of *N* unconnected, rigid bodies; it is assumed that the initial conditions of the bodies are known. The generalized displacement, or configuration, vector and the generalized velocity vector are denoted by $\mathbf{q}$ and $\dot{\mathbf{q}}$ respectively, where $\mathbf{q} = \left[(\mathbf{q}^1)^T \quad (\mathbf{q}^2)^T \quad \ldots \quad (\mathbf{q}^N)^T\right]^T$ and the vectors $\mathbf{q}^i$, $i = 1, 2, \ldots, N$ are the generalized coordinate *n*-vectors describing the configuration of the i^{th} body. Formulating a set of unconstrained equations of motion using Lagrangian (or Newtonian) mechanics, we then have

$$M(\mathbf{q},t)\,\ddot{\mathbf{q}} = \mathrm{F}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\,, \tag{1}$$

where $M(\mathbf{q},t)$ is a known $N \cdot n$ by $N \cdot n$ positive definite mass matrix, and $\mathbf{F}(\mathbf{q},\dot{\mathbf{q}},t)$ is a known $N \cdot n$ generalized force column vector with $N \cdot n$ elements. By 'known' here we mean that *M* and **F** are known functions of their respective arguments. We can then write the unconstrained acceleration for the system as (we drop the arguments for brevity, unless needed for clarity),

$$\mathbf{a}(\mathbf{q},\dot{\mathbf{q}},t) = M^{-1}\mathbf{F}\,. \tag{2}$$

It is now assumed that the system described in equation (1) is subjected to a set of *m* smooth, consistent, equality constraints of the form

$$\phi_k(\mathbf{q},t) = 0, \quad k = 1, 2, \ldots, h\,, \text{ and } \psi_k(\mathbf{q},\dot{\mathbf{q}},t) = 0, \quad k = h+1, h+2, \ldots, m\,, \tag{3}$$

which are the prescribed (desired) trajectory requirements. By differentiating these *m* equations with respect to time, we can obtain the constraint matrix equation for the *m* trajectory requirements given by

$$A(\mathbf{q},\dot{\mathbf{q}},t)\,\ddot{\mathbf{q}}(t) = \mathbf{b}(\mathbf{q},\dot{\mathbf{q}},t)\,, \tag{4}$$

where *A* is an *m* by $N \cdot n$ matrix and **b** is an *m*-vector. If the initial conditions are 'close to,' but do not exactly satisfy the trajectory requirements in equation (3), a control must be included to bring the state to satisfy them. This is achieved by stabilizing the system. The stabilization is accomplished by modifying equation (3) to

$$\ddot{\phi}_k + \alpha_k \dot{\phi}_k + \beta_k \phi_k = 0\,, \; k = 1,2,\ldots,h\,, \text{ and } \dot{\psi}_k + \gamma_k \psi = 0, \quad k = h+1, h+2, \ldots, m\,, \tag{5}$$

where $\alpha_k, \beta_k, \gamma_k > 0$, are suitably chosen parameters so that the fixed point of the system (5), which is given by $\dot{\phi}_k = \phi_k = \psi_k = 0$, is exponentially asymptotically stable. Thus, the system's trajectory will approach the desired trajectory as time progresses. It is important to note that the stabilized constraint equation (5) can again be expressed in the form of equation (4).

To accommodate the additional trajectory requirements (constraints) imposed on the system, an additional control force, $\mathbf{F}^C$, is required. This is the constraint force that is applied to the *N* bodies so that the required motion, which satisfies the trajectory requirements (constraints) given in equation (3) (or, equation (5)), is obtained. The equation of motion for the controlled system is now expressed as

$$M\ddot{\mathbf{q}} = \mathbf{F}(\mathbf{q},\dot{\mathbf{q}},t) + \mathbf{F}^C(\mathbf{q},\dot{\mathbf{q}},t)\,, \tag{6}$$

where $\mathbf{F}^C$ is the generalized control force required to satisfy the trajectory requirements. This force can be determined explicitly by [10, 11],

$$\mathbf{F}^C(t) = M^{1/2}(AM^{-1/2})^+(\mathbf{b} - A\mathbf{a})\,. \tag{7}$$

The '+' symbol denotes the Moore-Penrose Generalized Inverse, of the matrix $AM^{-1/2}$. The acceleration of the controlled system, so that it satisfies the trajectory requirements (constraints) given in equation (3) (or, equation (5)) is then

$$\ddot{\mathbf{q}} = \mathbf{a} + M^{-1/2}(AM^{-1/2})^{+}(\mathbf{b} - A\mathbf{a}). \tag{8}$$

Equation (8) is called the fundamental equation of motion [20] for constrained mechanical systems, and includes all the dynamics and control of the system necessary to satisfy the trajectory requirements described in equation (3) (or, equation (5)).

In this paper we propose to control a multi-body space system using the general method described above by defining trajectory requirements as a set of constraints. In order to properly model the connected, multi-body spacecraft system it is also required that constraints be imposed such that the two individual masses m_1 and m_2 behave as the single system that is shown in Fig. 1. In addition, to convey the generality of the method we model the system in orbit around a central body. Detailed discussion on modeling and control of the two-mass spacecraft system shown in Fig. 1 is presented in the next two sections.

3. EQUATIONS OF MOTION FOR A DUMBBELL SHAPED SPACECRAFT SYSTEM. The configuration space for the *2 uncoupled individual* masses m_i, $i = 1, 2$, of the spacecraft system consists of orbital and rigid body rotational coordinates (see Fig. 1). The position vector of the center of mass of the i^{th} mass with respect to the inertial coordinate system X-Y-Z is denoted by $\mathbf{r}^i = [x^i \;\; y^i \;\; z^i]^T$, $i = 1, 2$. We use quaternions to describe the rigid body rotational dynamics pertinent to each body so that large angle rotations of the system, as in tumbling, can be easily accommodated. Thus, along with the position vector $\mathbf{r}^i$ the i^{th} body will also have an associated quaternion *4*-vector $\mathbf{u}^i = \begin{bmatrix} u_0^i & u_1^i & u_2^i & u_3^i \end{bmatrix}^T$, $i = 1, 2$. The explicit *14*-vector specifying the configuration of the two mass system at any time *t* is then given by

$$\mathbf{s} := [(\mathbf{r}^1)^T \quad (\mathbf{r}^2)^T \quad (\mathbf{u}^1)^T \quad (\mathbf{u}^2)^T]^T. \tag{9}$$

In the following sub-sections we consider the orbital and rotational dynamics of each of the bodies. Then we demonstrate how, by using the fundamental equation of motion for constrained systems described above, we are able to couple the dynamics of the orbital and attitude motion along with an elastic potential between the bodies, to obtain the complete dynamical description of the two-body elastic dumbbell shaped system (shown in Fig. 1) in orbit around the planet.

3.1. Unconstrained Motion of Masses in Gravitational and Elastic Fields. The uniform gravitational potential function per unit mass of the i^{th} body about a spherical planet in an inertial frame whose origin is at the center of the planet is expressed as

$$V_i(x^i, y^i, z^i) = -\frac{\mu}{r^i}, \tag{10}$$

so that the acceleration and the force due to the gravity potential function for the i^{th} mass are then found as

$$\mathbf{a}_g^i = -\nabla V_i = -\begin{bmatrix} \dfrac{\partial V_i}{\partial x^i} & \dfrac{\partial V_i}{\partial y^i} & \dfrac{\partial V_i}{\partial z^i} \end{bmatrix}^T \text{ and } \mathbf{F}_g^i = diag(m_i, m_i, m_i)\,\mathbf{a}_g^i, \quad i = 1, 2. \tag{11}$$

We shall denote the gravitational force vector on the two masses by $\mathbf{F}_{\text{g}} := \left[(\mathbf{F}_g^1)^T,\ (\mathbf{F}_g^2)^T\right]^T$.

Denoting the relative distance between the two masses by $\Delta = \left| \mathbf{r}^2 - \mathbf{r}^1 \right|$, the components of the elastic forces between the masses are given by (see Fig. 1)

$$\mathbf{F}_e = \begin{bmatrix} F_{x^1} \\ F_{y^1} \\ F_{z^1} \\ F_{x^2} \\ F_{y^2} \\ F_{z^2} \end{bmatrix} = -\nabla U_e = -\frac{1}{\Delta}\left[K_l(\Delta - l_e) + K_{nl}(\Delta - l_e)^3 \right] \begin{bmatrix} -(x_2 - x_1) \\ -(y_2 - y_1) \\ -(z_2 - z_1) \\ x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}, \tag{12}$$

where K_l and K_{nl} are the spring constants related to the linear and the cubically nonlinear spring joining $\mathbf{C}_1$ to $\mathbf{C}_2$, respectively (see Fig. 1).

3.2. Unconstrained Attitude Motion. To formulate the unconstrained rotational dynamics we use Lagrange's equation. We use an extended angular velocity vector, $\boldsymbol{\omega}^i$, for the i^{th} mass that can be expressed in terms of the quaternion coordinates, $\mathbf{u}^i$, by the relation [21]

$$\boldsymbol{\omega}^i = 2{Q^i}^T \dot{\mathbf{u}}^i, \quad i = 1,\ 2, \tag{13}$$

where $\boldsymbol{\omega}^i = \left[0,\ \omega_1^i,\ \omega_2^i,\ \omega_3^i\right]^T$. The last three components of the extended angular velocity vector are simply the usual angular velocity components of the i^{th} body expressed in its body-fixed frame of reference. We shall take each of these body-fixed frames to be along the principal directions of inertia of each of the two bodies, as depicted in Fig. 1. The quaternion matrix Q^i is defined by

$$Q^i(\mathbf{u}^i) = \begin{bmatrix} u_0^i & -u_1^i & -u_2^i & -u_3^i \\ u_1^i & u_0^i & -u_3^i & u_2^i \\ u_2^i & u_3^i & u_0^i & -u_1^i \\ u_3^i & -u_2^i & u_1^i & u_0^i \end{bmatrix}. \tag{14}$$

The equation of motion, assuming that the four components of the unit quaternion are independent is then given by

$$M_u^i(\mathbf{u}^i)\ddot{\mathbf{u}}^i := [4Q^i I^i {Q^i}^T]\ddot{\mathbf{u}}^i = -8\dot{Q}^i I^i {Q^i}^T \dot{\mathbf{u}}^i - 4\left|\dot{\mathbf{u}}^i\right|\mathbf{u}^i := \widetilde{\mathbf{S}}_u^i(\mathbf{u}^i, \dot{\mathbf{u}}^i, t), \tag{15}$$

where the diagonal matrix $I^i = Diag(1, I_1^i, I_2^i, I_3^i)$. Thus, $\ddot{\mathbf{u}}^i = {M_u^i}^{-1}\widetilde{\mathbf{S}}_u^i := \mathbf{a}_u^i$.

We next enforce the constraint that $\mathbf{u}^i$ is a unit quaternion, so that

$${u_0^i}^2 + {u_1^i}^2 + {u_2^i}^2 + {u_3^i}^2 = 1, \quad i = 1, 2, \tag{16}$$

and, via equations (6) and (7), we get the equation of motion

$$M_u^i \ddot{\mathbf{u}}^i = \widetilde{\mathbf{S}}_u^i + {M_u^i}^{1/2}(A_u^i {M_u^i}^{-1/2})^+(\mathbf{b}_u^i - A_u^i \mathbf{a}_u^i) := \mathbf{S}_u^i(\mathbf{u}^i, \dot{\mathbf{u}}^i, t),\ i = 1, 2, \tag{17}$$

where the matrices A_u^i and $\mathbf{b}_u^i$ are appropriately obtained by differentiating equation (16) twice with respect to time. The unconstrained rotational equations of motion for the two masses can then be written as

$$M_u \ddot{\mathbf{u}} := \begin{bmatrix} M_u^1 \ddot{\mathbf{u}}^1 \\ M_u^2 \ddot{\mathbf{u}}^2 \end{bmatrix} := \mathbf{S}_u . \tag{18}$$

3.3. Coupled Orbital, Attitude, and Elastic Motions. Now that we have completely described the decoupled unconstrained motions of the two masses comprising the dumbbell shaped spacecraft, we can formulate the equations of motion for the coupled system. Combining equations (11), (12), and (17), we have the unconstrained equations of motion for the two-mass dumbbell shaped system moving in our gravitational and elastic potential as

$$M\ddot{\mathbf{s}} := M \begin{bmatrix} \ddot{\mathbf{r}}^1 \\ \ddot{\mathbf{r}}^2 \\ \ddot{\mathbf{u}}^1 \\ \ddot{\mathbf{u}}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{F_g} + \mathbf{F_e} \\ \mathbf{0}_{8x1} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{6x1} \\ \mathbf{S}_u \end{bmatrix} := \mathbf{F} , \tag{19}$$

where the positive definite block-diagonal matrix $M = diag(m_1, m_1, m_1, m_2, m_2, m_2, M_u^1, M_u^2)$.

The above equation still does not model the dynamics of our two-mass dumbbell spacecraft system. To model the connection between the two masses so that their relative motion is restricted only to a translation along the line $\mathbf{C}_1\, \mathbf{C}_2$ (see Fig. 1) whose direction is fixed relative to both the masses, we shall further require that:

(i) the body-fixed unit vectors $\boldsymbol{\eta}^1$ and $\boldsymbol{\eta}^2$ always point along the straight line $\mathbf{C}_1\, \mathbf{C}_2$, and
(ii) there is no relative rotational motion between the masses about the $\mathbf{C}_1\, \mathbf{C}_2$ axis.

We shall impose these requirements as *structural constraints* on the motion of the system described by equation (19). The first requirement can be written as (see Fig. 1),

$$\boldsymbol{\eta}^1 \times \mathbf{r}_{12} = \boldsymbol{\eta}^2 \times \mathbf{r}_{12} = 0 , \tag{20}$$

where $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ is the position vector between $\mathbf{C}_1$ and $\mathbf{C}_2$. Equation (20), which yields a set of 4 constraint equations, can be rewritten, in component form, as

$$\tilde{\eta}^1 E^1 \begin{bmatrix} (x^2 - x^1) \\ (y^2 - y^1) \\ (z^2 - z^1) \end{bmatrix} = \tilde{\eta}^2 E^2 \begin{bmatrix} (x^2 - x^1) \\ (y^2 - y^1) \\ (z^2 - z^1) \end{bmatrix} = 0 , \tag{21}$$

where E^i is the transformation matrix from the space-fixed coordinate frame to the body-fixed coordinate frame of the i^{th} mass in terms of quaternions [21], which is given by

$$E^i(\mathbf{u}^i) = \begin{bmatrix} ({u_0^i}^2 + {u_1^i}^2 - {u_2^i}^2 - {u_3^i}^2) & 2(u_1^i u_2^i + u_0^i u_3^i) & 2(u_1^i u_3^i - u_0^i u_2^i) \\ 2(u_1^i u_2^i - u_0^i u_3^i) & ({u_0^i}^2 - {u_1^i}^2 + {u_2^i}^2 - {u_3^i}^2) & 2(u_2^i u_3^i + u_0^i u_1^i) \\ 2(u_1^i u_3^i + u_0^i u_2^i) & 2(u_2^i u_3^i - u_0^i u_1^i) & ({u_0^i}^2 - {u_1^i}^2 - {u_2^i}^2 + {u_3^i}^2) \end{bmatrix} . \tag{22}$$

In equation (21), $\tilde{\eta}^i$ is the skew symmetric matrix representation of the vector $\boldsymbol{\eta}^i$ whose components are taken in the i^{th} body-frame of reference, as demanded by the cross product operation in equation (20). The second requirement can be expressed by the relation

$$\omega_1^1 = \omega_1^2 . \tag{23}$$

Thus, equation (21) along with equation (23) give the 5 necessary structural constraints that cause the two masses to act as one single (oscillating) dumbbell spacecraft system.

Differentiating these 5 structural constraints appropriately with respect to time, and writing them in the form of equation (5) where the stabilization parameters α_k, β_k and γ_k are appropriately selected, we can form our constraint matrix equation as given in equations (4) and (5). Using equation (7), the motion in three-dimensional space of the dumbbell-shaped spacecraft system in the planet's gravitational field is then written as

$$M\ddot{\mathbf{q}} = \mathbf{F} + M^{1/2}(A_s M^{-1/2})^{+}(\mathbf{b_s} - A_s\ddot{\mathbf{s}}) := \mathbf{F} + \mathbf{F_S} , \tag{24}$$

where $\ddot{\mathbf{s}}$, M and $\mathbf{F}$ are defined in equation (19). The 5 by 14 constraint matrix, A_s, and the 5-vector $\mathbf{b_s}$ in equation (24) are both associated with the structural constraints. The force $\mathbf{F_S}$ that needs to be applied to the system (described by equation (19)) in order to enforce them is given by equation (7). Thus, equation (24) gives the equation that describes the *free motion* of the dumbbell spacecraft system in our gravitational and elastic potential field. The acceleration of the dumbbell spacecraft system, using equation (24), is given by

$$\mathbf{a_S} = M^{-1}(\mathbf{F} + \mathbf{F_S}) . \tag{25}$$

4. TUMBLING TRAJECTORY CONTROL. In this section we demonstrate the effectiveness of this formulation by considering the planar motion of the dumbbell spacecraft system in a polar orbit in the X-Z plane around Mars, whose gravitational field is assumed to be uniform. We find the control needed for the vibrating dumbbell system so that it precisely tumbles as a given function of its orbital position.

Using our free motion description of the dumbbell spacecraft system in equation (24), we now impose a tumbling trajectory requirement on the system, so that its tumbling is a function of its orbital position. The tumbling requirement we impose (see Fig. 2) is

$$\theta(t) = \chi_1(t)\, z^1(t) + \chi_2(t)\, z^2(t) , \tag{26}$$

where $\chi_1(t) = \dfrac{2\pi}{r_{cm}(t)}\left(\dfrac{m_1}{m_1+m_2}\right)$, $\chi_2(t) = \dfrac{2\pi}{r_{cm}(t)}\left(\dfrac{m_2}{m_1+m_2}\right)$, and $r_{cm}(t)$ is the distance to the center of mass of the spacecraft system from the origin of the space-fixed coordinate system (see Fig. 2). The angle $\theta(t)$ given in equation (26) is the rotation angle of the dumbbell system about the $\boldsymbol{\gamma}^i$ -axes, i = 1,2, where the unit vectors $\boldsymbol{\gamma}^i$ point into the plane of the paper along the space-fixed Y-direction described by the vector $\mathbf{e} = [0 \quad 1 \quad 0]$. This constraint is such that the dumbbell spacecraft exactly completes a full tumble ($\theta : 0 \to 2\pi$) in the inertial frame of reference every quarter orbit around Mars as shown in Fig. 2.

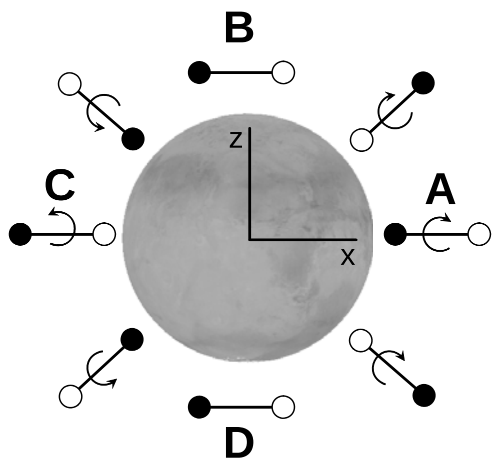

Fig. 2. The initial conditions are prescribed when the dumbbell system is at position **A**. As it moves counterclockwise in its orbital motion around Mars, it tumbles in the clockwise direction till it reaches position **B,** completing a tumble through 2π radians. It then tumbles in the opposite direction until it comes to position **D**, after which it reverses its tumbling direction until it reaches position **B** again. Mass m_1 is shown in blue, m_2 in red.

The tumble is such that it changes tumbling direction every half orbit. The space-fixed rotation axis **e** and the rotation $\theta(t)$ are related to the quaternion that describes the rotation of mass m_1 by the four constraint equations

$$\phi_1 := u_0^1(t) - \cos\frac{\theta(t)}{2} = 0\,, \qquad \phi_2 := u_1^1(t) = 0\,, \tag{27}$$

and,

$$\phi_3 := u_2^1(t) - \sin\frac{\theta(t)}{2} = 0\,, \qquad \phi_4 := u_3^1(t) = 0\,. \tag{28}$$

Equations (27) and (28) comprise what we shall call the tumbling trajectory requirements. Note that for $t > 0$ we need to apply these tumbling requirements to only one of the masses (here, the mass m_1) in the spacecraft system, since our structural constraints ensure that the two masses act as one dumbbell system. To properly apply these trajectory requirements to the dumbbell system we must apply our structural constraints (given by equations (21) and (23)) along with the desired tumbling trajectory requirements. This ensures that the control force being computed to satisfy the trajectory requirements continues to satisfy our structural constraints. Differentiating equations (21), (23), (27), and (28) appropriately with respect to time we can write these constraint equations in the form of equation (5), as before. Thus, the trajectory constraint matrix $A_{\mathbf{T}}$ and the vector $\mathbf{b_T}$ are generated as in equation (4). Using the unconstrained acceleration in equation (25), equation (7) explicitly gives the additional generalized force $\mathbf{F_T}$ needed to be applied to the system so that it satisfies the desired tumbling trajectory. We now add this trajectory control force to the dumbbell spacecraft system described by equation (24) to obtain its equation of motion as

$$M\ddot{\mathbf{q}} = \mathbf{F} + \mathbf{F}_{\mathrm{S}} + \mathbf{F}_{\mathrm{T}}\,. \tag{29}$$

From the generalized force, $\mathbf{F}_{\mathrm{T}}$ **,** we extract the control forces and the torques acting on masses m_1 and m_2 about their respective body-fixed axes.

The masses and principal inertia tensors comprising the dumbbell spacecraft (see Fig. 1) are taken to be $m_1 = 1000$ kg, $m_2 = 800$ kg, $I^1 = diag(13000, 10000, 7000)$ kg-m^2, and $I^2 = diag(8000, 6000, 11000)$ kg-m^2. The linear and nonlinear stiffness constants of the spring are selected as $K_l = 1$ and $K_{nl} = 0.1$. The distance between the two masses when the spring between them is unstretched is taken to be $l_e = 10$m. The gravitational constant used for Mars is $\mu_{Mars} = 4.2828380415705753 \times 10^4$ km^3/s^2 (see equation (10)).

The initial conditions are chosen so that mass m_1 is at an altitude of $l^1 = 600$ km above Mars's mean equatorial radius R (= 3397km) and mass m_2 is at an altitude of $l^2 = l^1 + l_e$, where $l_e = 10$ m, so that the spring between the two masses is initially unstretched. The $\boldsymbol{\eta}^i$ axes of both of the masses initially lie along the space-fixed X-direction (see Fig. 2). The initial orbit velocities are chosen so that mass m_1 is in a circular orbit and mass m_2 has an initial velocity such that the system satisfies all the structural constraints and the trajectory tumbling requirements. The initial orbital conditions are then

$$\mathbf{r}^1(0) = \begin{bmatrix} R + l^1 & 0 & 0 \end{bmatrix} \text{km}, \quad \mathbf{r}^2(0) = \begin{bmatrix} R + l^2 & 0 & 0 \end{bmatrix} \text{km}, \quad \text{and,} \tag{30}$$

$$\dot{\mathbf{r}}^1(0) = \begin{bmatrix} 0 & 0 & \sqrt{\dfrac{\mu_{Mars}}{R + l^1}} \end{bmatrix} \text{km/s}, \quad \dot{\mathbf{r}}^2(0) = \begin{bmatrix} 0 & 0 & \dfrac{1 - l_e \chi_1(0)}{1 + l_e \chi_2(0)} \dot{z}_1(0) \end{bmatrix} \text{km/s}. \tag{31}$$

The initial rotation angle $\theta(0) = 0$, and the angular velocity $\dot{\theta}(0) = \dfrac{\dot{z}^1(0) - \dot{z}^2(0)}{l_e}$, so that by equations (27) and (28), we get

$$\mathbf{u}^1(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{u}^2(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and,} \tag{32}$$

$$\dot{\mathbf{u}}^1(0) = \begin{bmatrix} 0 & 0 & \dfrac{\dot{\theta}(0)}{2} & 0 \end{bmatrix}, \quad \dot{\mathbf{u}}^2(0) = \begin{bmatrix} 0 & 0 & \dfrac{\dot{\theta}(0)}{2} & 0 \end{bmatrix}. \tag{33}$$

Equations (30)-(33) thus provide the necessary initial conditions for integrating the system of equations given in equation (29). Using the stabilization parameters $\alpha_k = 4$, $\beta_k = 8$, and $\gamma_k = 0.5$, this system of equations is integrated for approximately one orbit around Mars using MATLAB's ode113 with a relative error tolerance of 1×10^{-12} and an absolute error tolerance of 1×10^{-16}. Fig. 3 shows the motion of the

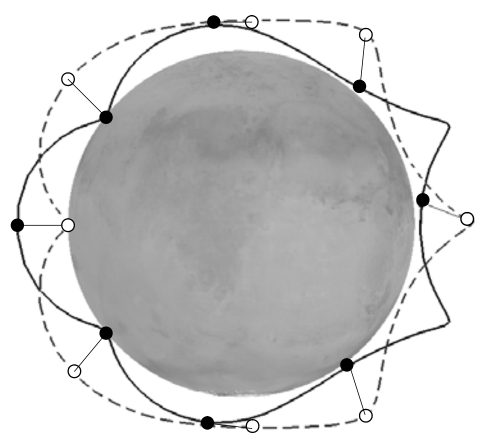

Fig. 3. The solid and dashed lines show the simulated trajectories (amplified to aid visualization) traced out by masses m_1 and m_2, respectively, of the dumbbell spacecraft system.

dumbbell system's trajectory for a full orbit, which is approximately 8000s. The errors in meeting the trajectory requirements (equations (27) and (28)) are shown in Fig. 4. We note that these errors are of the same order of magnitude as the tolerances used in numerically integrating equation (29). Fig. 5 shows the vibratory motion of the orbiting dumbbell system as it precisely tumbles according to equation (26). The necessary generalized control forces extracted from $\mathbf{F}_\mathrm{T}$, which are required to be applied to masses m_1 and m_2 in order to precisely maintain the tumbling trajectory (equation (26)), are plotted in Figs. 6 and 7. They consist of both forces and body torques. Several of the generalized forces are seen to be essentially zero, being of the same order of magnitude as the error tolerances used in the integration of equation (29).

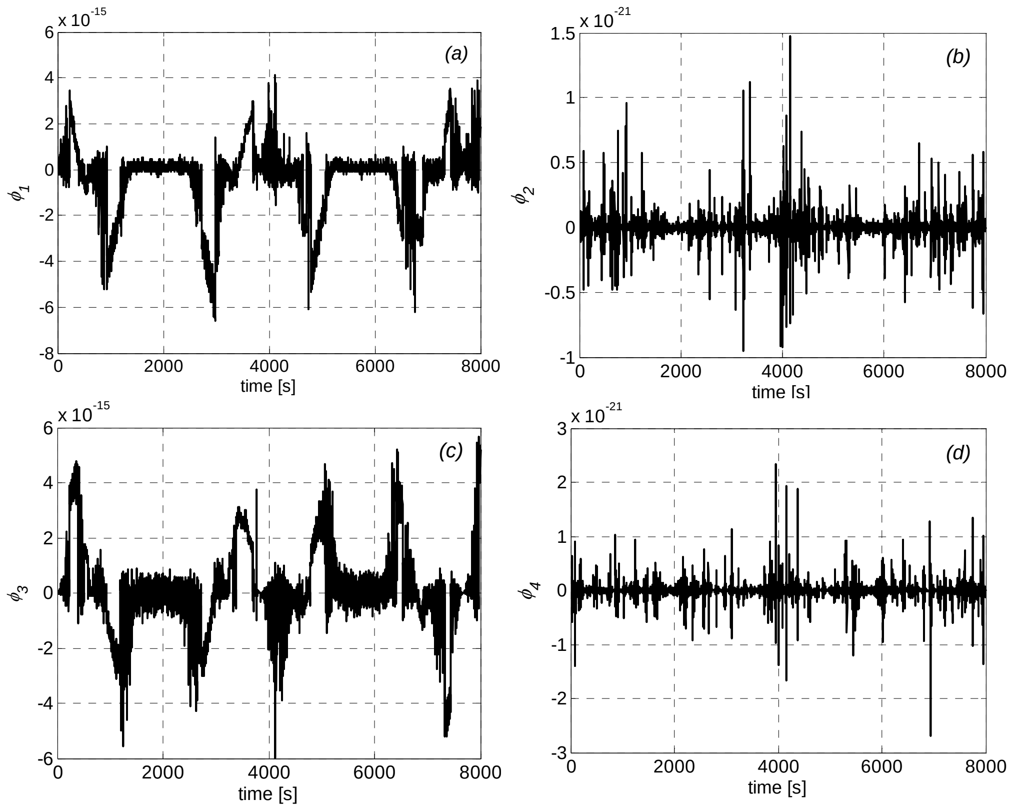

Fig. 4. (*a*, *b*, *c*, *d*) Error in trajectory constraints given by equations (26) and (27)**.**

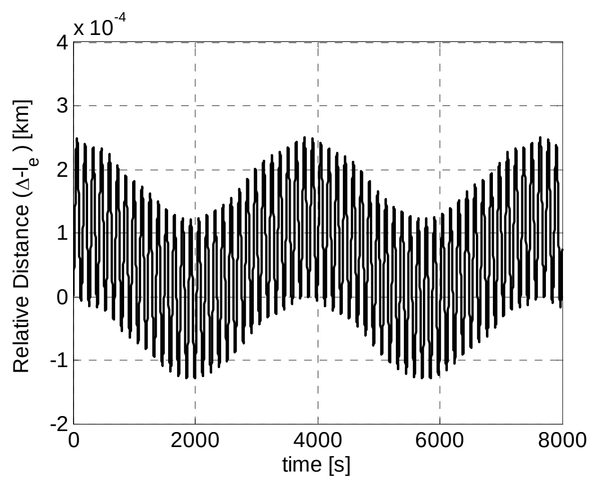

Fig. 5. Relative motion between the two masses showing oscillations of the dumbbell-shaped spacecraft system as it tumbles in orbit according to equation (26).

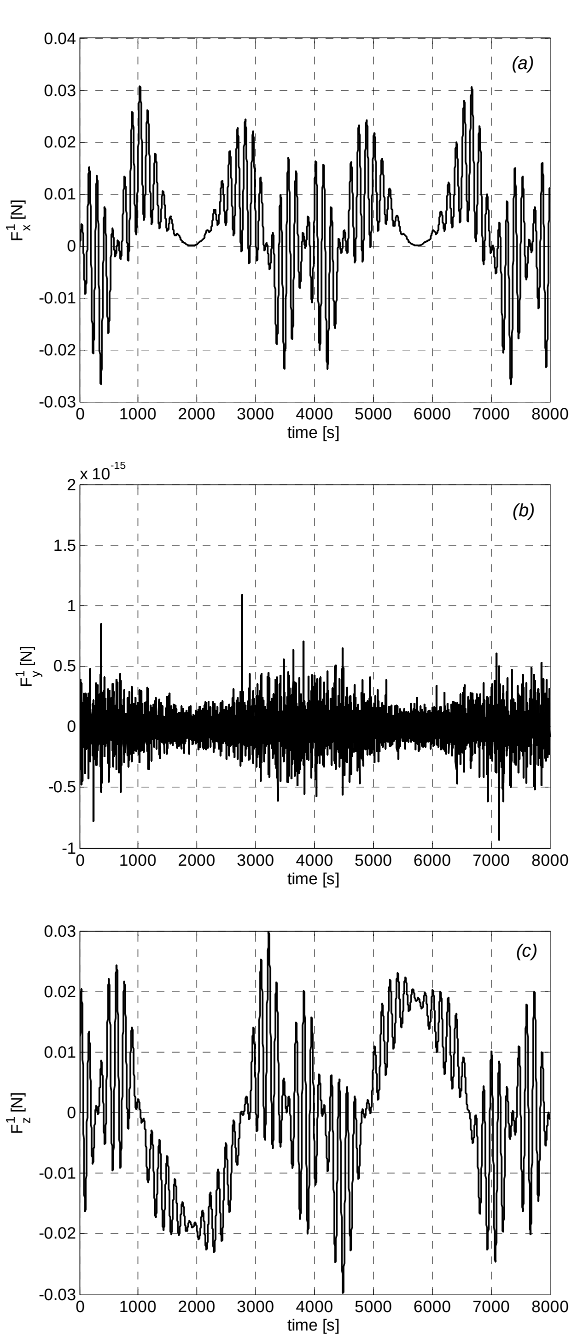

Fig. 6 (*a*, *b*, *c*). Trajectory control forces in the X-Y-Z directions on mass m_1.

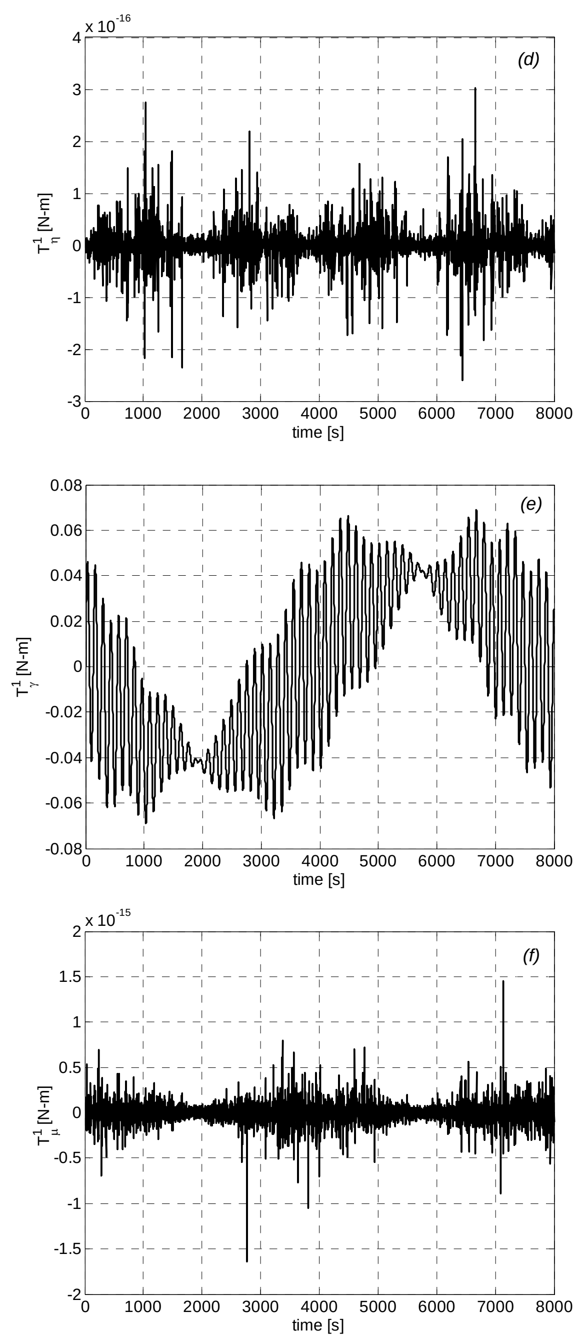

Fig. 6 (*d*, *e*, *f*). Trajectory control torques about the $\boldsymbol{\eta}^1-\boldsymbol{\gamma}^1-\boldsymbol{\mu}^1$ body-fixed axes on mass m_1.

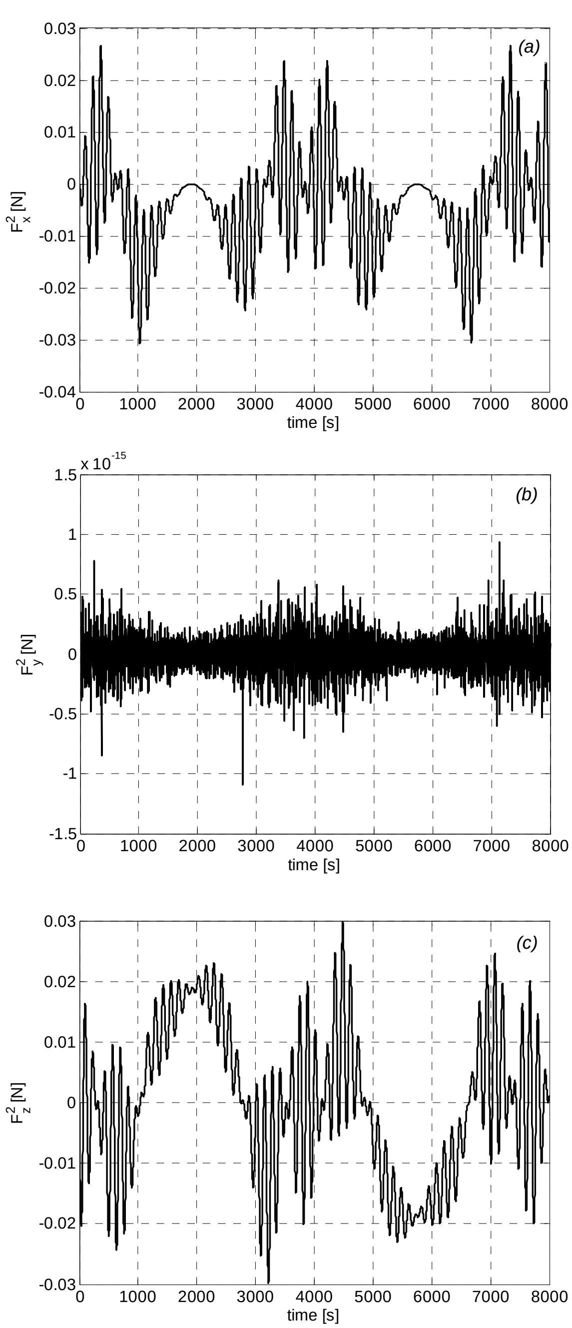

Fig. 7 (*a*, *b*, *c*). Trajectory control forces in the X-Y-Z directions on mass m_2.

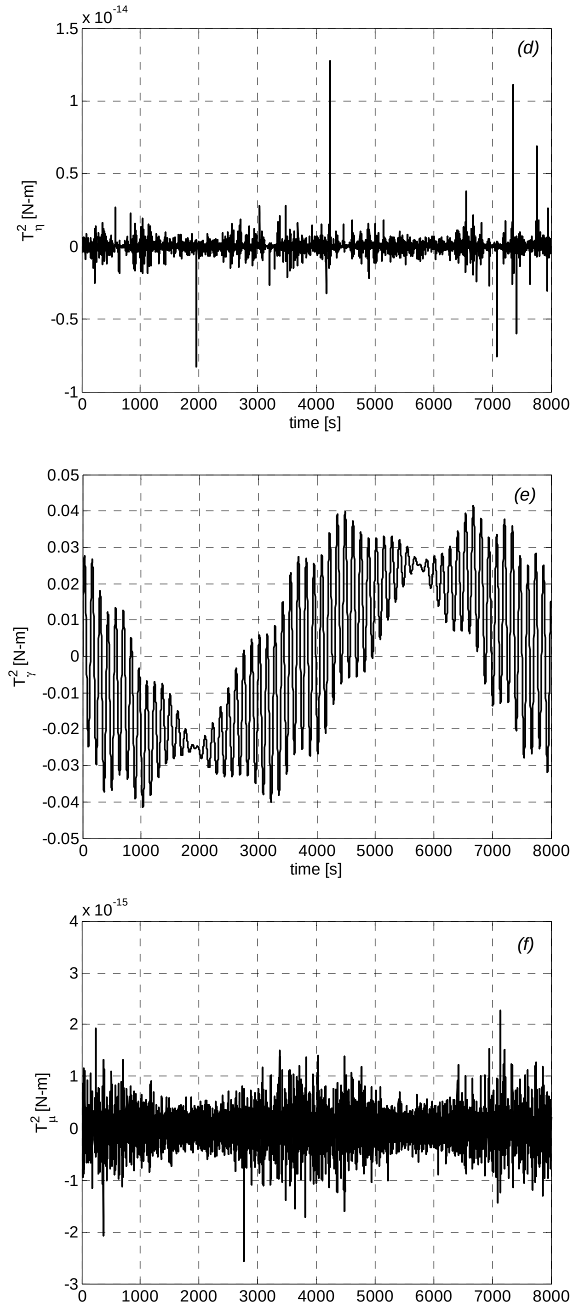

Fig. 7 (*d*, *e*, *f*). Trajectory control torques about the $\boldsymbol{\eta}^2-\boldsymbol{\gamma}^2-\boldsymbol{\mu}^2$ body-fixed axes on mass m_2.

5. CONCLUSIONS. In this paper we have developed a new and novel methodology for finding, in closed form, the control forces needed for a multi-body dumbbell-shaped spacecraft system to precisely execute a tumbling trajectory as it orbits around a planet. The coupled vibrational, attitudinal, and orbital motions are considered in their entirety without any linearizations or approximations, and the generalized control forces are found, again, without making any approximations. The procedure relies on the fundamental equation of motion, and utilizes the central idea that trajectory requirements can be recast as a set of constraints on the orbiting system. The control forces are found explicitly, in closed form, for the nonlinear system, and they can be determined in real time. The methodology is shown to give good results, and the precision with which the trajectory requirements are satisfied is commensurate with the precision with which the numerical integration of the equation of motion is carried out. The approach has the advantage of being easy to implement. It points the way towards the development of new and simple methods for the exact control of highly nonlinear, complex, multi-body systems.

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