Formation Flight of Multiple Rigid Body Spacecraft

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The control of multiple rigid body spacecraft in formation flight is explored in this paper. We propose a general control methodology based on the general equations of motion for constrained mechanical systems to solve the problem of formation flying with attitude requirements. Examples of formation flying rigid body spacecraft with various attitude requirements will be given about an oblate central body. The examples demonstrate the broad applicability and scalability of the method. The approach leads to a simple non-linear control methodology that allows exact control of both orbital and attitude dynamics. No linearization of the dynamics is required. The approach used reveals a systematic technique for controlling complex non-linear systems.

Nomenclature

A	=	constraint matrix
a	=	unconstrained acceleration vector
a_p	=	second-order perturbation acceleration due to spherical harmonics
$C_{n,m}$	=	tesseral harmonics coefficients
E	=	body-fixed to space-fixed rotation matrix
$\mathbf{e}$	=	Euler axis
G	=	universal gravitational constant
J	=	inertia matrix
J_n	=	zonal harmonics coefficients
K	=	rotational kinetic energy
M	=	mass matrix
P	=	body frame torques
P_{q}	=	generalized quaternion torques
Q	=	quaternion matrix
q	=	generalized rotational state coordinates
R	=	body-centered reference radius
r	=	geocentric distance
s	=	generalized coordinates
S	=	generalized forces
S^C	=	constraint forces
$S_{n,m}$	=	sectorial harmonics coefficients
T	=	kinetic energy

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U	=	gravitational potential function
V	=	rotational potential energy
α	=	modal damping
β	=	natural frequency
φ	=	kinematic constraints
ϕ	=	latitude of the spacecraft
θ	=	rotation about Euler axis
η	=	body-axis vector
λ	=	longitude of the spacecraft
μ	=	gravitation constant (GM)
ω	=	extended angular velocity vector

I. Introduction

THE concept of spacecraft formation flight allows us to build a virtual system in which many spacecraft may together operate as one large spacecraft, or in which collaborative motions among various spacecraft can be accomplished. For many future space missions, a proper understanding of the interaction between orbital and attitude dynamics of formation flying spacecraft is essential so as to achieve the high level of control authority required to maintain precise formations and attitude requirements. Traditional methods of analyzing space systems are to decouple the local motion of the spacecraft (i.e., attitude motion) and that of the orbital motion.

Early formation flying researchers were focused on relative motion, interception, and docking control, and concern was mostly focused on the orbital dynamics and not the attitude dynamics. Many of these attempts in modeling the control problem were done by linearizing the dynamics about a reference orbit. In the case of the Clohessy-Wiltshire equations[1], where an ideal non-perturbed circular orbit is assumed as the reference orbit, the relative motion of bodies (particles) was analyzed about the reference orbit. Recent work by Kristiansen[2], Wang[3], and other authors[4-6], incorporates attitude modeling and control to the formation flying problem. For example, in Ref. 2 the Clohessy-Wiltshire equations were used along with attitude motion for a full 6-DOF model. The design and implementation of a state feedback linearizing controller and passivity based controller was also included. Other commonly used control methods, such as the Lyapunov and LQR methods, also rely on linearization in order to arrive at a suitable control force.

This paper explores a control methodology for a spacecraft system that accounts for the orbital and attitude motions of the various individual spacecraft. The control approach we develop uses principles rooted in analytical dynamics and is a continuation of work done in Ref. 7, where the orbital motion was investigated, but not the attitude motion. The important contribution of this paper is that this methodology proposed herein does not require any linearization of the dynamics or the controls. This is accomplished by re-framing the required trajectory requirements for formation-flight-with-attitude-control as a problem of constrained motion. The constraints describe the dynamics of the desired formation, and/or tracking and attitude requirements, and we directly determine, in closed form, the exact explicit control forces necessary to maintain these constraints. The power of the method lies with our ability to couple and de-couple attitude and orbital motion when desired.

The control methodology employed in this paper was developed by Udwadia and Kalaba[8-10] for precisely controlling multi-body systems. Although they applied the method to simple mechanical systems, it has been shown that this general method can be easily applied to highly complex astrodynamical problems.[7,11] This paper generalizes this method by (a) applying the theory to the control of multiple rigid body spacecraft with desired orbital and attitude requirements moving around a spinning planet with a non-uniform gravity field, and (b) formulating a general control methodology for spacecraft attitude control in terms of quaternions, noting that the Lagrange equations of motion that correctly describe the necessary attitude dynamics require that these quaternions be always constrained to have unit norm.

Our general methodology is illustrated using two examples. The first aims at controlling a spacecraft in Mars' non-uniform gravitational field so that its orbit around the planet is confined to a plane while the craft simultaneously rotates in a prescribed time-varying manner about an axis that is fixed in an inertial frame of reference. The second example presents a multi-body attitude tracking problem. Here, we have two spacecraft moving around Mars in different orbits with one of them required to always point directly towards the other.

In the next section we discuss the general formulation of formation flight for any given trajectory and/or attitude motion requirement, which we re-frame in terms of constraints. Next, we derive the unconstrained equations of motion of a rigid body spacecraft, and then obtain the explicit and exact control forces required to satisfy the necessary orbital and attitude constraints. In the following section we provide two examples, and in the last section we summarize the paper.

II. General Formulation of Formation Flying

We begin the study of formation flying by considering N rigid bodies as shown in Figure 1. Let $\Im_I$ denote an inertial reference frame and $\Im_{B_i}$ denote the body-fixed reference frame for the i^{th} rigid body. The origin of the body-frame, $\Im_{B_i}$, is located at the center of mass of the i^{th} body, and the body-frame axes point along its principal axes of inertia.

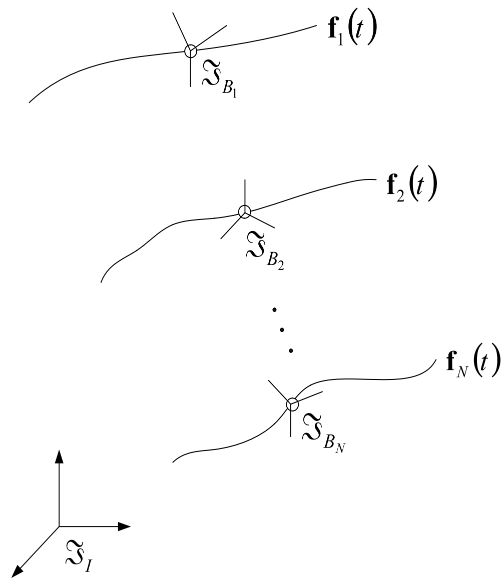

Figure 1. *N* rigid body trajectory description.

Denoting the n parameters needed to describe the configuration of the i^{th} rigid body by $s^i = \left[s_1^i, s_2^i, \ldots, s_n^i\right]^T$, we can formulate a set of unconstrained equations of motion for the system, using Lagrangian mechanics, by the relation

$$M(s,t)\ddot{s} = S(s,\dot{s},t), \tag{1}$$

where M is the $N \cdot n$ by $N \cdot n$ positive definite mass matrix, $s := [(s^1)^T, (s^2)^T, \ldots, (s^N)^T]^T$ and the $N \cdot n$-vector S contains the 'given' generalized forces. We can then write the unconstrained acceleration for the system by the $N \cdot n$-vector a, which is given by

$$\ddot{s} = M^{-1}S := a(s,\dot{s},t). \tag{2}$$

It is now assumed that the system of formation flying vehicles is subjected to m consistent equality constraints of the form

$$\varphi_k(s,\dot{s},t) = 0, k = 1, 2, \ldots, m, \tag{3}$$

which are just the prescribed (desired) time-varying flight-formation (including attitude) trajectory requirements. These requirements may be prescribed in terms of the $N \cdot n$ generalized coordinates, and/or time explicitly. The initial conditions $s_0 = s(t_0)$ and $\dot{s}_0 = \dot{s}(t_0)$ are assumed to satisfy the requirements in Eq. (3), so that $\varphi_k(s_0,\dot{s}_0,t_0) = 0, k = 1, 2, \ldots, m$. Thus, the trajectory requirements for formation flying are recast as additional constraints applied to the unconstrained dynamical system that is described by Eq. (1). Note that the m constraints in Eq. (3) include, among others, holonomic, non-holonomic, and scleronomic constraints. Assuming that Eq. (3) is sufficiently smooth upon differentiation with respect to time, it yields

$$A(s,\dot{s},t)\ddot{s} = b(s,\dot{s},t), \tag{4}$$

where A is an m by $N \cdot n$ matrix, and b is an m-vector. If the initial conditions do not satisfy the constraints in Eq. (3), we would need to stabilize the system to the constraint trajectory. For holonomic constraints we can write Eq. (3) in the form

$$\ddot{\varphi}_k + \alpha_k \dot{\varphi}_k + \beta_k \varphi_k = 0, k = 1,2,\ldots,m, \tag{5}$$

where $\alpha_k, \beta_k > 0$, so that, the fixed point m-vector $\varphi = \dot{\varphi} = 0$ is asymptotically stable.[13] Equation (5) can again be expressed in the form of Eq. (4).

The presence of the constraints in Eq. (4) imposes an additional generalized force on the system so that its equation of motion now becomes

$$M(s,t)\ddot{s} = S(s,\dot{s},t) + S^c(s,\dot{s},t). \tag{6}$$

The additional force S^c arises due to the imposed constraints of the form described in Eq. (4). It has been known for some time that given the constraint equations in the form of Eq. (4) we can obtain an explicit equation of motion that satisfies these constraints at *each* instant of time. The formula that governs the evolution of the constrained system is (for clarity, we drop the arguments of the various quantities)

$$M\ddot{s} = S + M^{1/2}\left(AM^{-1/2}\right)^{+}(b - Aa), \tag{7}$$

where the symbol '+' denotes the Moore-Penrose inverse of the matrix $AM^{-1/2}$ and a is defined in Eq. (2). We now have a system that enables us direct control of multiple rigid body vehicles in any desired formation pre-defined by the constraints in Eq. (4). From Eq. (7) we see that the exact, explicit, closed form, control force is given by

$$S^c(s,\dot{s},t) = M^{1/2}\left(AM^{-1/2}\right)^{+}(b - Aa). \tag{8}$$

This simple yet comprehensive approach to constrained motion, originally introduced by Udwadia and Kalaba,[8] enables us to easily impart the required formation motion to the N vehicle ensemble.

III. Spacecraft Formation Flying Equations of Motion

Consider the problem of controlling multiple spacecraft, modeled herein as rigid bodies, so that they move in a precise formation around a central body as shown in Figure 2. The configuration space for the i^{th} spacecraft with mass $m_i, i = 1,2,\ldots,N$ and inertia matrix $J^i, i = 1,2,\ldots,N$ consists of orbital and rigid body rotational coordinates. The position vector $\mathbf{r}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^T$ is the coordinate of the center of mass of the i^{th} spacecraft with respect to the space-fixed, inertial coordinate frame located at the center of the central body. To develop the rigid body rotational dynamics pertinent to each spacecraft we use quaternions so that arbitrary orientations can be realized without the presence of any singularities. Thus, along with the position vector $\mathbf{r}_i$ the i^{th} spacecraft will also have an associated quaternion $q^i = \begin{bmatrix} q_0^i & q_1^i & q_2^i & q_3^i \end{bmatrix}$. We now define the generalized displacement vector

$$s^i = \begin{bmatrix} x_i & y_i & z_i & q_0^i & q_1^i & q_2^i & q_3^i \end{bmatrix}^T := [(\mathbf{r}_i)^T, (q^i)^T]^T, i = 1, 2, \ldots, N \tag{9}$$

that describes the position and orientation of the i^{th} formation flying spacecraft. In the following sub-sections we will consider the orbital and rotational dynamics using this framework.

A. Unconstrained Orbital Motion

The orbital motion of each of N different spacecraft is expressed by the well known "two-body problem" as

$$m_i \ddot{\mathbf{r}}_i = -\frac{\mu m_i}{r_i^3}\mathbf{r}_i + m_i a_p^i, i = 1, 2, \ldots, N , \tag{10}$$

where $r_i = |\mathbf{r}_i|$ and the gravitational interaction between the spacecraft is ignored. The gravitational parameter $\mu = GM_\mu$ denotes the product of the gravitational constant and the mass of the central body. The additional acceleration term a_p in Eq. (10) includes perturbation accelerations caused by non-uniform gravitational effects of the central body, and when $a_p = 0$, the equation reverts to the Newtonian relation for a uniform spherical central body.

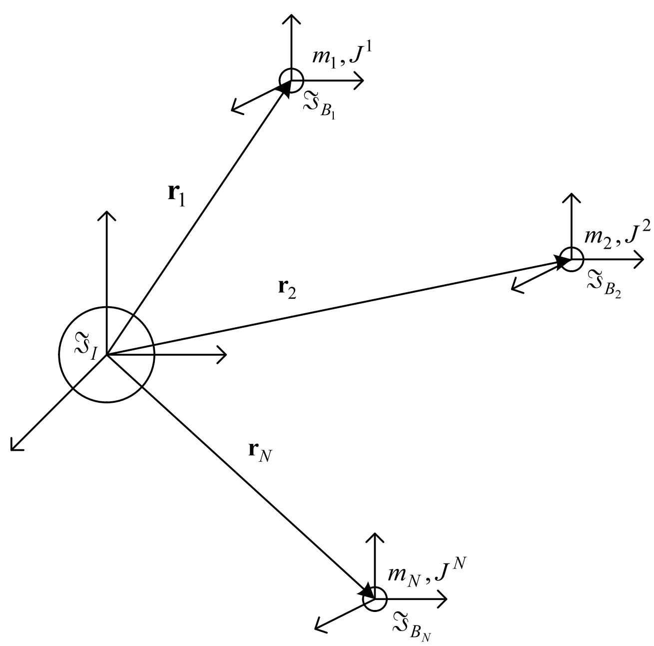

Figure 2. Spacecraft formation flying description

The unconstrained acceleration given by Eq. (10) can be expressed in terms of the gradient of a potential function

$$\ddot{\mathbf{r}}_i = \nabla U_i, \quad i = 1, 2, \ldots, N \tag{11}$$

where U is derived from the potential function of a spheroid, which is given by

$$U_i = -\frac{\mu}{r_i} + \frac{\mu}{r_i}\sum_{n=2}^{\infty} J_n \left(\frac{R_i}{r_i}\right)^n P_n(\sin\phi_i) - \frac{\mu}{r_i}\sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_i}{r_i}\right)^n P_n^m(\sin\phi_i)(C_{n,m}\cos m\lambda_i + S_{n,m}\sin m\lambda_i). \tag{12}$$

The terms J_n, $C_{n,m}$, and $S_{n,m}$ are the coefficients associated with zonal, tesseral, and sectorial harmonics of the central body.[12] From Eqs. (11) and (12) we now have the unconstrained acceleration of the i^{th} spacecraft orbiting an oblate body under a non-uniform gravitational field.

B. Unconstrained Rotational Motion

Analysis in rigid body dynamics is typically carried out utilizing Euler's rotational equations, which describe the rotation of a rigid body in a frame of reference fixed in the rotating body. As mentioned earlier, we shall consider the rigid body rotational dynamics of each of the *N* different spacecraft in terms of quaternions. This provides a way of realizing arbitrary orientations and attitude dynamics with no inherent geometric singularities or complexities due to trigonometric relations, as is the case with Euler angles.

We begin by formulating the unconstrained rotational dynamics using Lagrange's equation. Corresponding to the i^{th} spacecraft there exists an extended angular velocity vector, ω^i, which can be expressed in terms of the quaternion coordinates of the body, by the relation

$$\omega^i = 2Q^{i^T}\dot{q}^i, i = 1,2,\ldots,N, \tag{13}$$

where $\omega^i = [0,\ \omega_1^i,\ \omega_2^i,\ \omega_3^i]^T$.[16] The last three components of our extended angular velocity vector are simply the usual angular velocity components of the i^{th} body expressed in the body-fixed frame of reference. The quaternion matrix *Q* results from the product of two quaternions and is defined by

$$Q^i\left(q^i\right) = \begin{bmatrix} q_0^i & -q_1^i & -q_2^i & -q_3^i \\ q_1^i & q_0^i & -q_3^i & q_2^i \\ q_2^i & q_3^i & q_0^i & -q_1^i \\ q_3^i & -q_2^i & q_1^i & q_0^i \end{bmatrix}. \tag{14}$$

To formulate the unconstrained equations of motion for rotational motion we use Lagrange's equation, which is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = P_q^i. \tag{15}$$

where P_q^i is the 'given' generalized force vector for the i^{th} spacecraft. In what follows, we shall assume toque free motion so that $P_q^i = 0$. The kinetic energy is simply the rotational energy given by

$$T^i = \frac{1}{2}\omega^{i^T}\hat{J}^i\omega^i. \tag{16}$$

Recalling Eq. (13), the inertia matrix $\hat{J}^i$ has the form,

$$\hat{J}^i = \begin{bmatrix} 1 & 0 \\ 0 & J^i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_1^i & 0 & 0 \\ 0 & 0 & J_2^i & 0 \\ 0 & 0 & 0 & J_3^i \end{bmatrix}. \tag{17}$$

where J_1^i, J_2^i, J_3^i are the principal moments of inertia along the 1-, 2- and 3- principal directions in the i^{th} body. Using Eq. (13) with Eq. (16) the kinetic energy for the i^{th} body becomes

$$T^i = 2\dot{q}^{i^T} Q^i \hat{J}^i Q^{i^T} \dot{q}^i = 2q^{i^T} \dot{Q}^i \hat{J}^i \dot{Q}^{i^T} q^i . \tag{18}$$

Applying Eq. (15) *under the assumption that each of the four components of the quaternion* q^i *are independent*, we obtain Lagrange's equations for the i^{th} body as

$$4Q^i \hat{J}^i Q^{i^T} \ddot{q}^i + 8\dot{Q}^i \hat{J}^i Q^{i^T} \dot{q}^i + 4Q^i \hat{J}^i \dot{Q}^{i^T} \dot{q}^i = 0, i = 1,2,\ldots,N , \tag{19}$$

which we shall write, for notational convenience, as

$$M_q^i \left(q^i\right)\ddot{q}^i := [4Q^i \hat{J}^i Q^{i^T}]\ddot{q}^i = -8\dot{Q}^i \hat{J}^i Q^{i^T} \dot{q}^i - 4Q^i \hat{J}^i \dot{Q}^{i^T} \dot{q}^i := S_q^i \left(q^i, \dot{q}^i, t\right) i = 1,2,\ldots,N . \tag{20}$$

However, equation (20) is *not* yet the correct equation of motion since in actuality the components of the quaternion q^i are not all independent as assumed by us in obtaining it. In order to represent a rotation, q^i must be a *unit* quaternion. Hence,

$$q_0^{i^2} + q_1^{i^2} + q_2^{i^2} + q_3^{i^2} = 1, i = 1, 2,\ldots, N . \tag{21}$$

In other words, Eq. (21) is a constraint on the formulation in Eq. (20). Utilizing the result from Section II we can apply this constraint to Eq. (20) to obtain the true unconstrained rotational equations of motion for the i^{th} body. Recasting Eq. (20) into the form of Eq. (1) we have

$$M_q^i \left(q^i\right)\ddot{q}^i = S_q^i \left(q^i, \dot{q}^i, t\right) i = 1, 2,\ldots, N , \tag{22}$$

or equivalently

$$\ddot{q}^i = M_q^{i^{-1}} S_q^i = a_q^i . \tag{23}$$

After differentiating, the constraint in Eq. (21) is written in the form of Eq. (4) by

$$\begin{bmatrix} q_0^i & q_1^i & q_2^i & q_3^i \end{bmatrix} \begin{bmatrix} \ddot{q}_0^i \\ \ddot{q}_1^i \\ \ddot{q}_2^i \\ \ddot{q}_3^i \end{bmatrix} = -\dot{q}_0^{i^2} - \dot{q}_1^{i^2} - q_2^{i^2} - \dot{q}_3^{i^2} , \tag{24}$$

where we define the constraint matrix

$$A_q^i = \begin{bmatrix} q_0^i & q_1^i & q_2^i & q_3^i \end{bmatrix} \tag{25}$$

and resulting 1-vector $b_q^i = -\dot{q}_0^{i^2} - \dot{q}_1^{i^2} - q_2^{i^2} - \dot{q}_3^{i^2}$. Using Eq. (7), we can now explicitly write the true unconstrained rotational equations of motion in terms of quaternions for the torque free i^{th} body as

$$M_q^i \ddot{q}^i = S_q^i + M_q^{i^{1/2}} \left(A_q^i M_q^{i^{-1/2}} \right)^+ \left(b_q^i - A_q^i a_q^i\right) := \tilde{S}_q^i (q^i, \dot{q}^i, t) \tag{26}$$

or,

$$\ddot{q}^i = a_q^i + M_q^{i^{-1/2}}\left(A_q^i M_q^{i^{-1/2}}\right)^+ \left(b_q^i - A_q^i a_q^i\right). \tag{27}$$

C. *N* Formation-Flying Spacecraft Equations of Motion

With the orbital and rigid body dynamics described by Eq. (11) and Eq. (26), we can now consider the entire ensemble of N formation-flying spacecraft as a single dynamical system whose motion is then described by the $7N$ set of equations

$$M(\mathbf{s})\ddot{\mathbf{s}} = S(\mathbf{s},\dot{\mathbf{s}},t) , \tag{28}$$

where the $7N$ vector $\mathbf{s} = \left[\left(s^1\right)^T, \left(s^2\right)^T, \ldots, \left(s^N\right)^T\right]^T$ and the matrix M is a block-diagonal matrix containing N blocks, each block being a 7 by 7 matrix. The unconstrained acceleration of the entire system is

$$\ddot{\mathbf{s}} = M(\mathbf{s})^{-1} S(\mathbf{s},\dot{\mathbf{s}},t) = \mathbf{a}(\mathbf{s},\dot{\mathbf{s}},t). \tag{29}$$

From Eq. (10) and Eq. (25) we see that the i^{th} diagonal block of M is the block diagonal matrix $diag\left(m_i, m_i, m_i, 4Q^i \hat{J}^i Q^{i^T}\right)$. The $7N$ vector S in Eq. (28) is similarly written as a set of N different 7-component sub-vectors. Each sub-vector pertains to a single spacecraft, the i^{th} sub-vector is given by the 7-vector $\left[m_i \frac{\partial U_i}{\partial x_i} \quad m_i \frac{\partial U_i}{\partial y_i} \quad m_i \frac{\partial U_i}{\partial z_i} \quad \widetilde{S}_q^i\left(q^i, \dot{q}^i, t\right)\right]$, where $\widetilde{S}_q^i$ is defined in Eq. (26) and U_i is given by the potential function in Eq. (12).

The desired N spacecraft formation is now expressible by the constraints in Eq. (5) and we can write the formation constraint equation by

$$A(\mathbf{s},\dot{\mathbf{s}},t)\ddot{\mathbf{s}} = \mathbf{b}(\mathbf{s},\dot{\mathbf{s}},t). \tag{30}$$

The control force that is required to maintain this formation is explicitly given by Eq. (8) and re-written in terms of our N formation-flying spacecraft by

$$S^c(\mathbf{s},\dot{\mathbf{s}},t) = M^{1/2}\left(AM^{-1/2}\right)^+(\mathbf{b} - A\mathbf{a}). \tag{31}$$

The equations of motions that completely define the orbital and attitude dynamics for each formation flying spacecraft are then

$$M(\mathbf{s})\ddot{\mathbf{s}} = S(\mathbf{s},\dot{\mathbf{s}},t) + M^{1/2}\left(AM^{-1/2}\right)^+(\mathbf{b} - A\mathbf{a}). \tag{32}$$

This formulation allows us to easily describe and compute coupled or decoupled dynamics between orbital and attitude motion, and it explicitly provides the proper control forces needed for the system to execute required trajectory requirements.

IV. Numerical Examples

In the following section we will demonstrate the capability of this new methodology by presenting two classes of problems. The first example consists of a spacecraft (N = 1) required to: (a) orbit in a plane around a central body which exerts a non-uniform gravitational field on it, and (b) rotate about a space-fixed axis in a prescribed manner, which is given by a specified function of time. We then present a multibody spacecraft (N = 2) attitude tracking problem.

A. Example 1

Consider a system wherein a spacecraft is initially in a circular orbit at an altitude of 603 km above Mars. The spacecraft is modeled as a rectangular prism with $m = 100$ kg and $J = diag(77.08 \quad 93.75 \quad 20.83)$kg-m^2. We select the initial orbital conditions such that if a uniform gravitational field is assumed then the spacecraft will follow a circular orbit in the equatorial plane of Mars. The initial orbital conditions are

$$\mathbf{r}(0) = [R_{mars} + 603 \quad 0 \quad 0]\text{km} \tag{33}$$

and

$$\dot{\mathbf{r}}(0) = \left[0 \quad \sqrt{\frac{\mu_{mars}}{R_{mars} + 603}} \quad 0\right] \text{km/s}, \tag{34}$$

where $R_{mars} = 3397$ km is the radius of Mars, and $\mu_{mars} = 4.2828 \times 10^4$ km^3/s^2 is the gravitational parameter of Mars. If we use the non-uniform gravitational potential of Mars up to J_4 for the zonals and including the C_{21}, C_{22}, S_{21}, and S_{22} harmonic terms, we can expect that the orbit will not truly be a circular in-plane orbit. To demonstrate the methodology described in this paper, we maintain the spacecraft in its initial orbital plane such that the perturbations due to the non-uniform gravitational field are counteracted. Given the initial conditions in Eq. (33) and Eq. (34) we can see that the constraint

$$\varphi_1 = z(t) = 0\,, \tag{35}$$

where the z-axis is along the axis of rotation of Mars, must always be satisfied to maintain the spacecraft's orbit in the equatorial plane of Mars.

Additionally, we want to prescribe the spacecraft to rotate about a space-fixed axis **e** by the relation

$$\theta(t) = \theta_0 + d\sin(wt), \tag{36}$$

where the unit Euler axis $\mathbf{e} = [e_1 \quad e_2 \quad e_e]$ and rotation θ are related to the spacecraft quaternion by the four constraint equations

$$\varphi_2 = q_0 - \cos\frac{\theta}{2} = 0 \tag{37}$$

$$\varphi_3 = q_1 - e_1\sin\frac{\theta}{2} = 0 \tag{38}$$

$$\varphi_4 = q_2 - e_2\sin\frac{\theta}{2} = 0 \tag{39}$$

$$\varphi_5 = q_3 - e_3\sin\frac{\theta}{2} = 0\,. \tag{40}$$

Equations (37-40) are otherwise known as Euler symmetric parameters and are introduced in Ref. 14. We now have five constraint equations that completely define our desired formation, which is depicted in Figure 3. We select the rotation parameters of the formation as $w = \frac{2\pi}{1000}\frac{rad}{s}$, $d = 2\pi$, $\theta_0 = 1$, and the unit vector $\mathbf{e} = [0.5 \quad 0.5 \quad 0.7071]$.

Differentiating Eq. (35-40) we can write the constraint equations in the form of Eq. (5), and then as the constraint matrix equation in Eq. (4) by

$$\begin{bmatrix} 0_{5x2} & I_{5x5} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{q}_0 \\ \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} -\alpha_1 \dot{z} - \beta_1 z \\ -\alpha_2\left(\dot{q}_0 + \sin\left(\frac{\theta}{2}\right)\frac{\dot{\theta}}{2}\right) - \beta_2\left(q_0 - \cos\left(\frac{\theta}{2}\right)\right) - \cos\left(\frac{\theta}{2}\right)\frac{\dot{\theta}^2}{4} - \sin\left(\frac{\theta}{2}\right)\frac{\ddot{\theta}}{2} \\ -\alpha_3\left(\dot{q}_1 + \sin\left(\frac{\theta}{2}\right)\frac{\dot{\theta}}{2}\right) - \beta_3\left(q_1 - \cos\left(\frac{\theta}{2}\right)\right) - \cos\left(\frac{\theta}{2}\right)\frac{\dot{\theta}^2}{4} - \sin\left(\frac{\theta}{2}\right)\frac{\ddot{\theta}}{2} \\ -\alpha_4\left(\dot{q}_2 + \sin\left(\frac{\theta}{2}\right)\frac{\dot{\theta}}{2}\right) - \beta_4\left(q_2 - \cos\left(\frac{\theta}{2}\right)\right) - \cos\left(\frac{\theta}{2}\right)\frac{\dot{\theta}^2}{4} - \sin\left(\frac{\theta}{2}\right)\frac{\ddot{\theta}}{2} \\ -\alpha_5\left(\dot{q}_3 + \sin\left(\frac{\theta}{2}\right)\frac{\dot{\theta}}{2}\right) - \beta_5\left(q_3 - \cos\left(\frac{\theta}{2}\right)\right) - \cos\left(\frac{\theta}{2}\right)\frac{\dot{\theta}^2}{4} - \sin\left(\frac{\theta}{2}\right)\frac{\ddot{\theta}}{2} \end{bmatrix}, \tag{41}$$

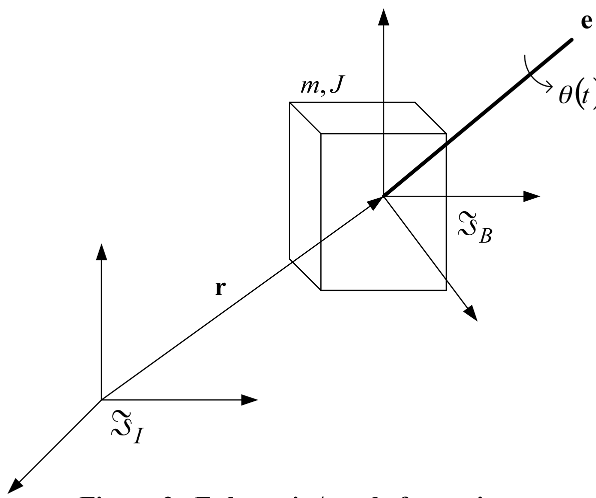

Figure 3. Euler axis / angle formation

where I denotes an identity matrix and the terms α_k, β_k are appropriately selected to stabilize the system to the constraint trajectory. For this example, we select the stabilization parameters as $\alpha_k = 30$ and $\beta_k = 100$. We now have the equations of motion for the desired formation by using our result in Eq. (32). Using the integrator ODE15s in MATLAB with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-9}, Figure 4 shows that with the orbit constraint the trajectory successfully maintains its initial orbital plane cancelling the perturbations that cause the spacecraft to leave the equatorial plane. We can see that the constraints in Eq. (35) and Eq. (37-40) are satisfied to machine precision in Figure 5. The required forces and torques in the body-frame necessary to maintain the orbital and attitude motion are shown in Figure 6. Although it is not derived here, we can relate the generalized quaternion torque P_q^i applied to the i^{th} spacecraft and given by Eq. (31) to the torque $P^i = \begin{bmatrix} P_1^i & P_2^i & P_3^i \end{bmatrix}^T$ about the body frame axes of the i^{th} spacecraft by comparing the formulation in Eq. (26) with

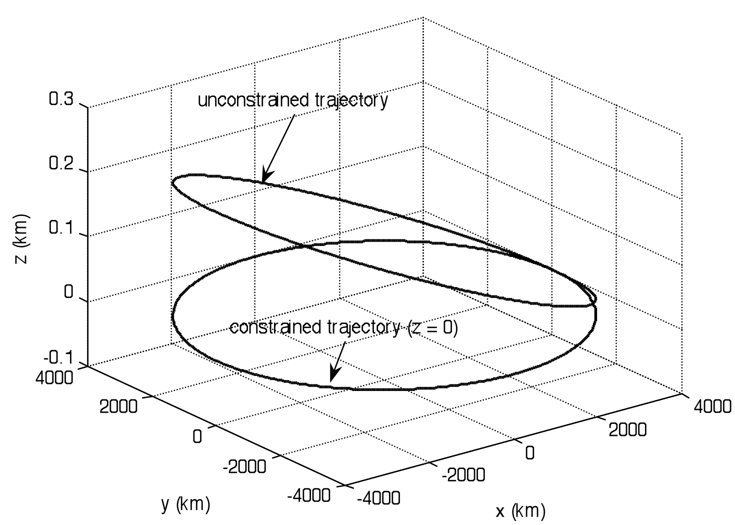

Figure 4. Example 1 orbit trajectory

a similar formulation utilizing Euler's rotational equations. The relation is a rather simple transformation and is given by

$$\begin{bmatrix} 0 \\ P^i \end{bmatrix} = \frac{1}{2} Q^T P_q^i, i = 1, 2, \ldots, N . \tag{42}$$

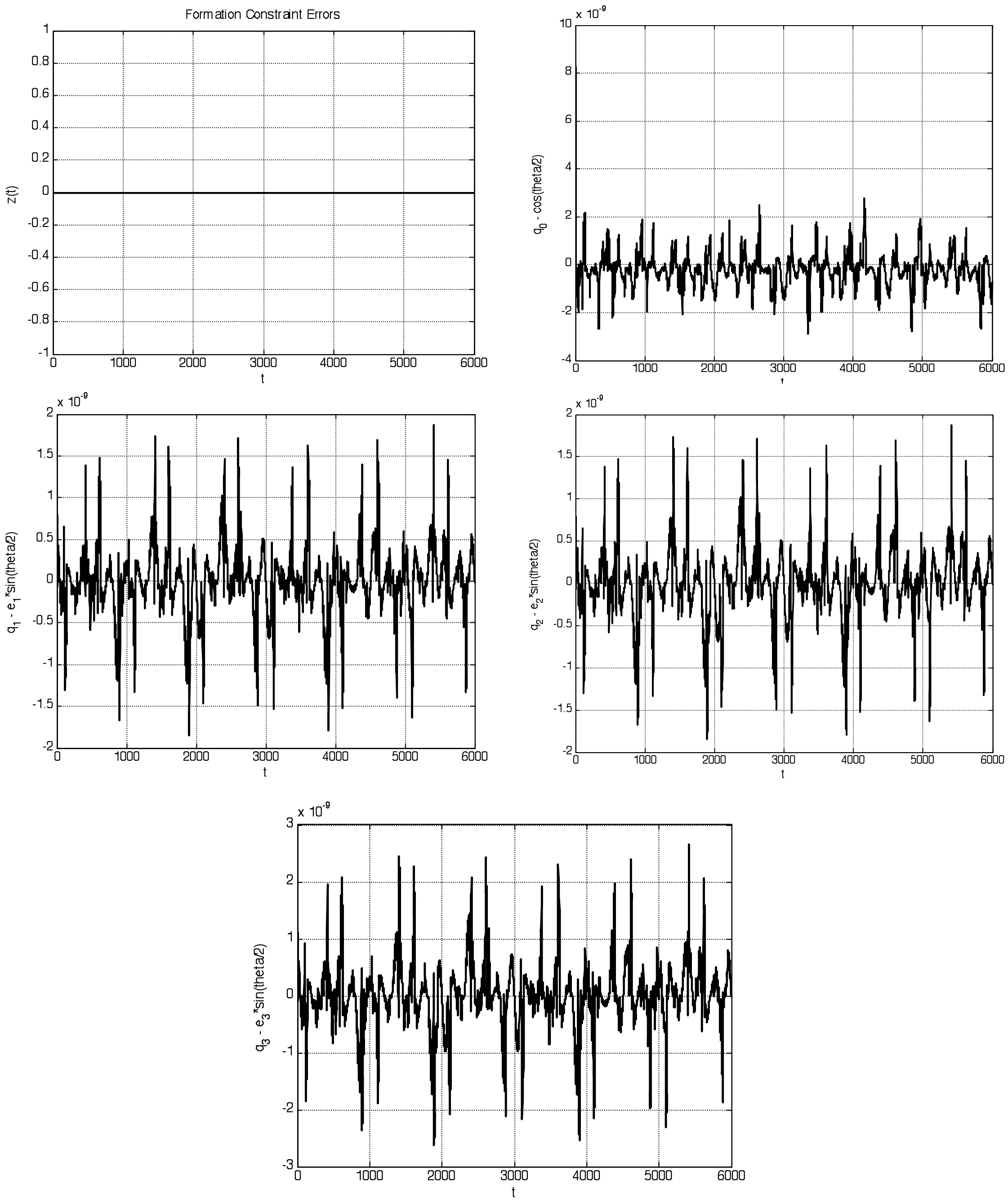

Figure 5. Constraint errors for Example 1, time in seconds.

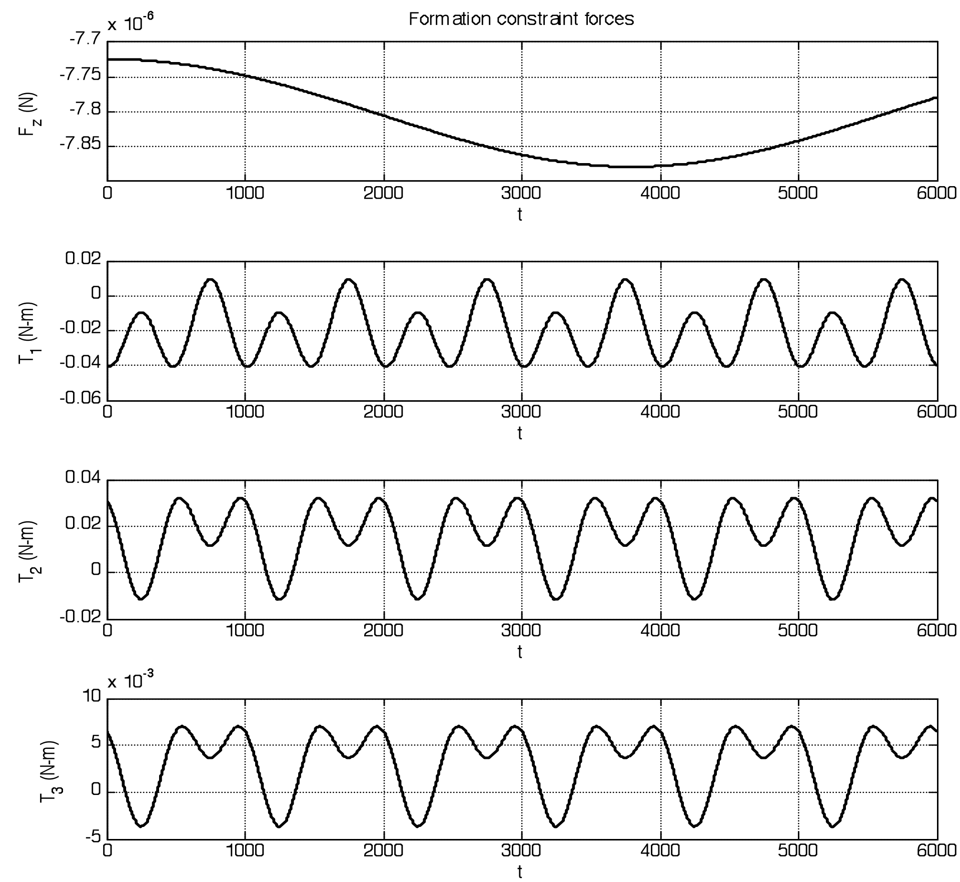

Figure 6. Constraint forces for Example 1, time in seconds.

B. Example 2

Consider the spacecraft attitude tracking problem wherein it is required that a spacecraft constantly points a desired body axis at another spacecraft, when both spacecraft are in different orbits around a central body that exerts a non-uniform gravitational attraction on each of them. Both spacecraft are allowed to exist in any arbitrary orbit. For simplicity, we select the body axis $\eta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ of the pointing spacecraft $\Im_{B_1}$ to point directly at the target spacecraft $\Im_{B_2}$ as shown in Figure 7. We model the pointing spacecraft as we did in Example 1 with $m_1 = 100\,\text{kg}$ and $J^1 = diag\begin{pmatrix} 77.08 & 93.75 & 20.83 \end{pmatrix}$ kg-m^2. The target spacecraft, $\Im_{B_2}$, is free to orbit with no constraints imposed, which means we only need to know its mass $m_2 = 100\,\text{kg}$. Similar to Example 1, we select the initial orbital conditions such that the two spacecraft are initially in a circular orbit if a uniform gravitational field is assumed. The pointing spacecraft is at an altitude of 603 km in the equatorial plane and the target spacecraft is at an altitude of 803 km in a polar orbit around Mars. The initial orbital conditions are then

$$\mathbf{r}_1(0) = \begin{bmatrix} R_{mars} + 603 & 0 & 0 \end{bmatrix}\text{km} \tag{43}$$

$$\mathbf{r}_2(0) = \begin{bmatrix} 0 & R_{mars} + 803 & 0 \end{bmatrix}\text{km} \tag{44}$$

$$\dot{\mathbf{r}}_1(0) = \begin{bmatrix} 0 & \sqrt{\dfrac{\mu_{mars}}{R_{mars}+603}} & 0 \end{bmatrix} \text{km/s} \tag{45}$$

$$\dot{\mathbf{r}}_2(0) = \begin{bmatrix} 0 & 0 & \sqrt{\dfrac{\mu_{mars}}{R_{mars}+803}} \end{bmatrix} \text{km/s}. \tag{46}$$

For the pointing spacecraft to successfully point at the target spacecraft, we must have the pointing axis η exactly aligned with the position vector $\mathbf{r}_{12}$, where $\mathbf{r}_{12} = [x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1]$. We must first rotate the position vector $\mathbf{r}_{12}$ into the body-fixed frame $\Im_{B_1}$. The body-fixed to space-fixed rotation parameterized by quaternions can be written by the rotation matrix

$$E(q^1) = E(q) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{47}$$

Rotation matrices in terms of quaternions are discussed in more detail in Ref. 15. Therefore, the desired pointing axis η is constrained to point along the position vector $\mathbf{r}_{12}$ by the constraint equation

$$\eta \times E^T \mathbf{r}_{12} = 0, \tag{48}$$

which yields the two constraint equations

$$\varphi_1 = 2(q_1q_3 + q_0q_2)(x_1 - x_2) + 2(q_2q_3 - q_0q_1)(y_1 - y_2) + (q_0^2 - q_1^2 - q_2^2 + q_3^2)(z_1 - z_2) = 0 \tag{50}$$

$$\varphi_2 = 2(q_1q_2 - q_0q_3)(x_2 - x_1) + (q_0^2 - q_1^2 + q_2^2 - q_3^2)(y_2 - y_1) + 2(q_2q_3 + q_0q_1)(z_2 - z_1) = 0. \tag{51}$$

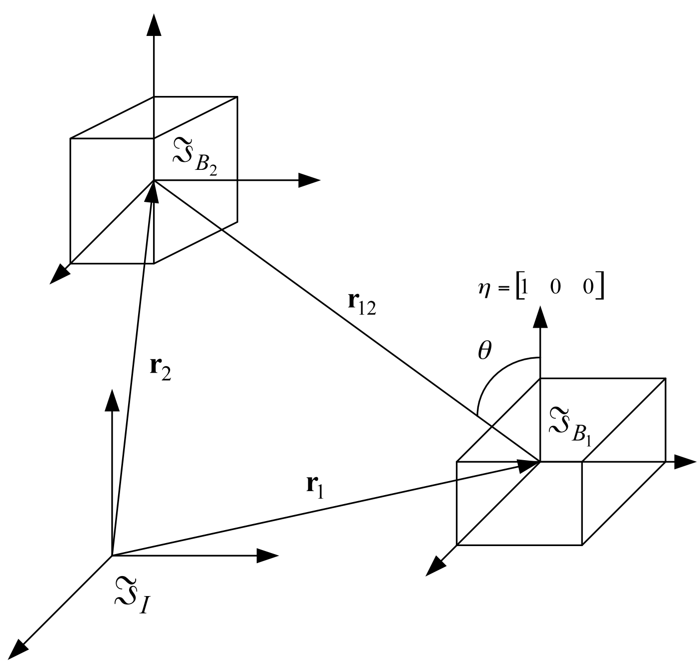

Figure 7. Attitude tracking formation

Note that we do not need any attitude information from the target spacecraft as we are only constraining the attitude of the pointing spacecraft. However, the orientation of the pointing spacecraft is a function of both the position coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$. Differentiating Eqs. (50)-(51) and writing them in the form of Eq. (5) we can formulate our constraint matrix equation by using Eq. (30). The equations of motion for the entire system are then given by our result in Eq. (32).

We select the initial attitude conditions of the pointing spacecraft so that they do not satisfy the constraint equations in Eq. (50-51). Using Euler parameters we start slightly off the trajectory by the initial attitude conditions

$$q(0) = \left[\cos\left(\frac{\theta_0}{2}\right) \quad e_1 \sin\left(\frac{\theta_0}{2}\right) \quad e_2 \sin\left(\frac{\theta_0}{2}\right) \quad e_3 \sin\left(\frac{\theta_0}{2}\right) \right], \tag{52}$$

where $\mathbf{e} = [0.5 \quad 0.5 \quad 0.7071]$ and $\theta_0 = \dfrac{170\pi}{180}$. To approach the constraint trajectory we select the stabilization parameters as $\alpha_k = 3$ and $\beta_k = 1$. Integrating the system with ODE15s in MATLAB with a relative error tolerance of 10^{-10} and an absolute error tolerance of 10^{-12}, Figure 8 shows the constraint errors defined by Eq. (50-51). Because the pointing spacecraft does not start on the correct 'attitude trajectory' we can see that the errors are not initially zero, but approach zero according to the stabilization method in Eq. (5).

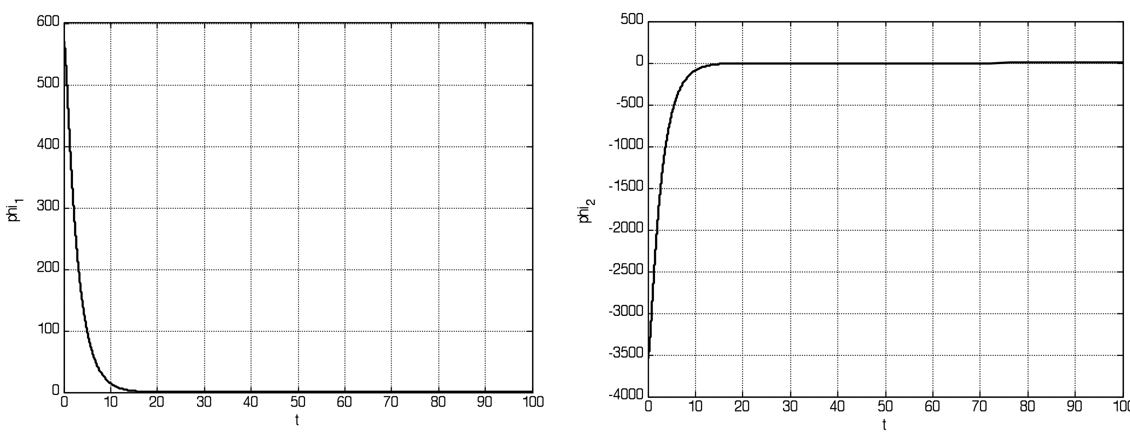

Figure 8. Constraint errors for Example 2, time in seconds.

V. Conclusion

This paper demonstrated the control of multiple rigid body spacecraft in formation flight. A general control methodology is employed based on the general equations of motion for constrained systems. Both orbital and attitude dynamics are simultaneously considered, and the singularity-free Lagrangian equations of motion using quaternions for expressing rotations are obtained. Using the general methodology, explicit, closed form expressions for the generalized control forces for satisfying both orbital and attitudinal trajectories are obtained. An example of a rigid body spacecraft orbiting an oblate rotating Mars in a fixed plane, with a prescribed attitudinal trajectory is used to demonstrate the ease and accuracy with which the methodology can be used. A second example involving a multi-body attitude tracking problem is also considered; a satellite orbiting an oblate rotating Mars is required to point at each instant of time towards another satellite in a very different orbit. Because no linearization or approximations to the nonlinear equations are carried out in finding the generalized control forces, we note from the numerical simulations the great precision with which the control methodology works. The examples demonstrate the simplicity with which the approach can be applied to complex formation flying problems and point to its broad range of applicability and scalability for the precision control of other more complex dynamical systems.

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